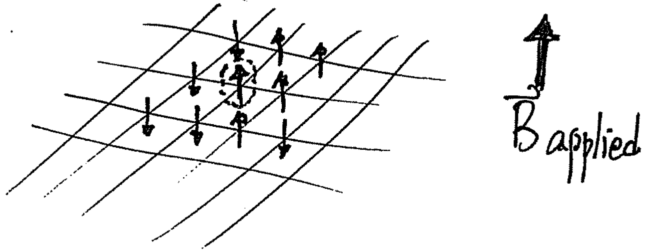
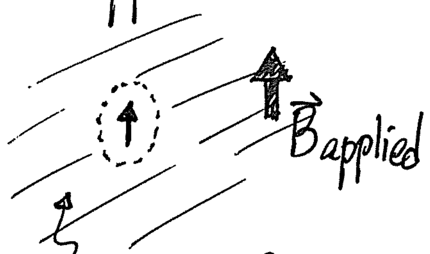
F. Simplest Mean-Field Theory: The Physical Idea



(↑) "feels" a field due to (interaction with neighboring spins + $\vec{B}_{applied}$)

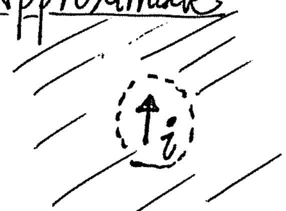
(i) Approximate as



some mean field due to surrounding spins

(↑) is under the influence of $\vec{B}_{local} = \underbrace{\vec{B}_{mean\,field}}_{\boxed{\text{yet-to-be-determined}}} + \vec{B}_{applied}$

(ii) Approximate



same mean field on all the spins

Argument: (↑) is nothing special

Deeper: Ignore differences in actual local fields at different spins

"ignore fluctuations"

[better approximation when there are more nearest neighbors]

(iii) Self-Consistency Condition

- Mean-field $= Jz\langle S\rangle \propto \underbrace{\langle S\rangle}_{\text{to-be-determined}} \Rightarrow$ better alignment gives a strong mean field
- But a strong mean field is needed to get at better alignment

$$\boxed{\begin{array}{l}\text{Alignment as quantified by } \langle S\rangle \text{ gives mean field and thus } B_{local} \\ \text{Mean field (thus } B_{local}\text{) gives } \langle S\rangle \text{ for quantifying Alignment}\end{array}}$$

$\therefore$ $\langle S\rangle$ must be determined self-consistently!

$$E(\{S_i\}) = -J\sum_{\langle ij\rangle} S_i S_j - B\sum_i S_i$$ [interaction S_iS_j term hard to treat]

$$\approx -J\sum_{\langle ij\rangle} S_i \underbrace{\langle S_j\rangle} - B\sum_i S_i$$ [approximation]

stat. mech. average of neighboring spin gives <u>mean field</u> $\langle S_j\rangle = \langle S\rangle$

$$\approx -J\langle S\rangle z\sum_i S_i - B\sum_i S_i$$ [become problem of <u>independent spins</u> in a field, as in paramagnetism]

to-be-determined # nearest neighbors

["$\dfrac{J\langle s\rangle z}{\mu_B}$" is the strength of mean field]

$$\approx -\underbrace{(J\langle s\rangle z + B)}_{\text{"a field acting on spin } i\text{"}}\sum_i S_i \qquad (2)$$

Mean-field external applied field (if present)

z = coordination number of lattice (e.g. $z=4$ for square lattice)

"Copy" Paramagnetic result in a clever way

$$E(\{S_i\}) \approx -(J\langle S\rangle z + B)\sum_i S_i \qquad (\text{recall: } S_i = +1, -1)$$

Paramagnetism

$$\langle \mu_z \rangle = \mu_B \tanh\left(\frac{\mu_B B}{kT}\right)$$

for one $J=\frac{1}{2}$ dipole in a field B

"Copy" results:

$$\boxed{\langle S\rangle = \tanh\left[\beta J z\langle S\rangle + \beta B\right]} \quad (3)$$

Mean-field equation for $\langle S\rangle(T)$

a self-consistency equation for $\langle S\rangle$

- As defined, $\langle S\rangle$ is a number between -1 to $+1$.
- $\langle S\rangle$ is the stat. mech. average of S_z when system is at temp. T
- $\langle S\rangle$ is related to $\langle \mu_z\rangle$ by $\mu_B\langle S\rangle \Rightarrow \langle S\rangle$ is the "average magnetisation per spin" (in units of μ_B) (call it "m")
- Writing $\langle S\rangle$ as m, Eq. (3) is

$$\boxed{m = \tanh\left[\beta J z m + \beta B\right]} \quad (4)$$

Does mean-field theory give Spontaneous Magnetization?

- Set $B = 0$ (no applied field), Eq. (4) is

$$m = \tanh\left(\frac{Jzm}{kT}\right) \quad (5)$$

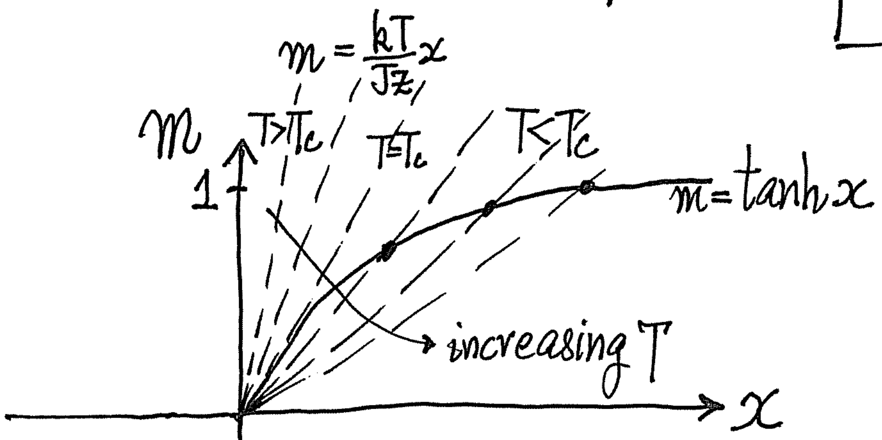
$\langle S \rangle = 0$ is obviously a solution!

$\boxed{\text{Q: Any solutions of } \langle S \rangle \neq 0 \text{ for finite } T\text{?}}$

Think graphically

- Call $x = \frac{Jz}{kT} m \Rightarrow \boxed{m = \frac{kT}{Jz} x}$ and Eq. (5) gives $\boxed{m = \tanh x}$

two curves



Similarly down here!

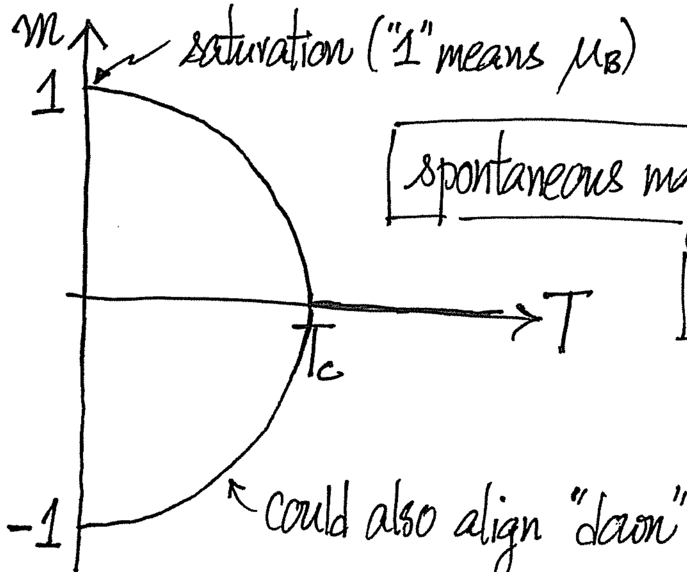
$T > T_c$: intersects at $m = 0$ only

$T = T_c$: last temperature that intersects at $m = 0$ only

$T < T_c$: $m \neq 0$ solutions arise

$T \to 0$: $m \to 1$ (saturation for one spin)

$\Rightarrow$



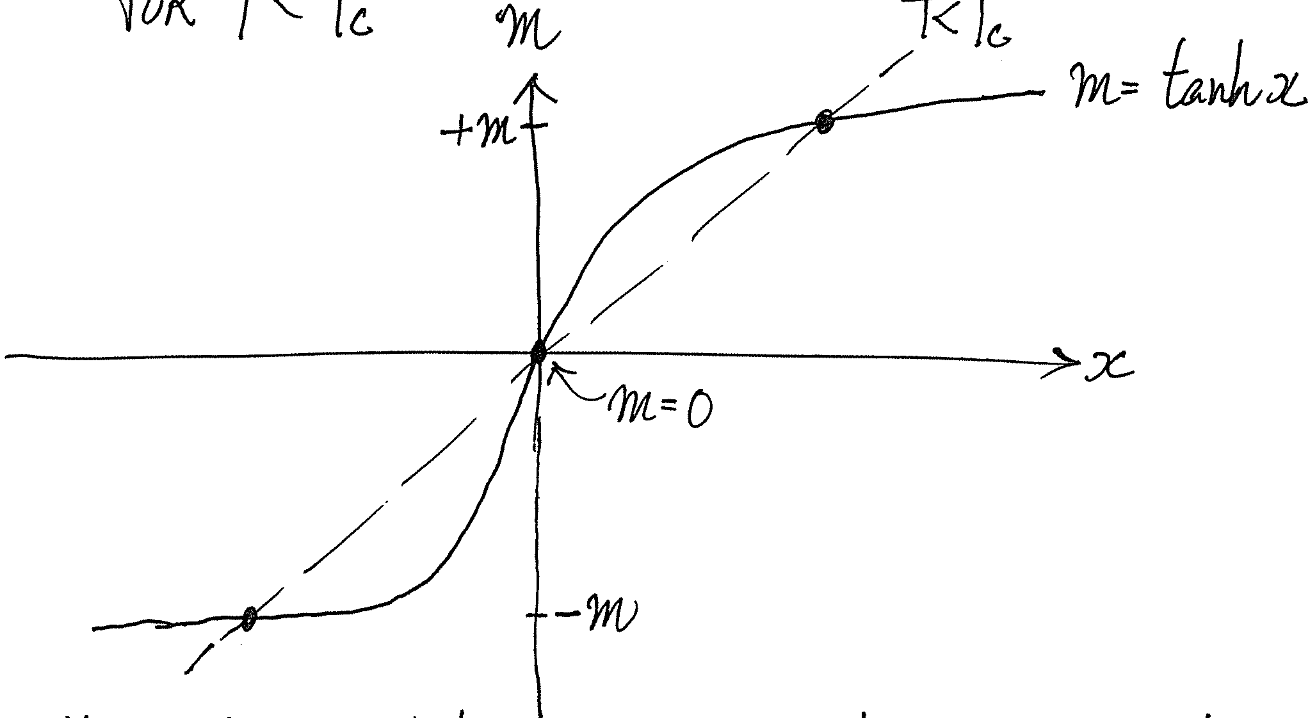
$\boxed{\text{spontaneous magnetization}}$

$\boxed{B_{applied} = 0}$

Formally,

$$m = \tanh\left(\frac{Jzm}{kT}\right) \text{ allows } \underline{3 \text{ solutions}}$$

for $T < T_c$

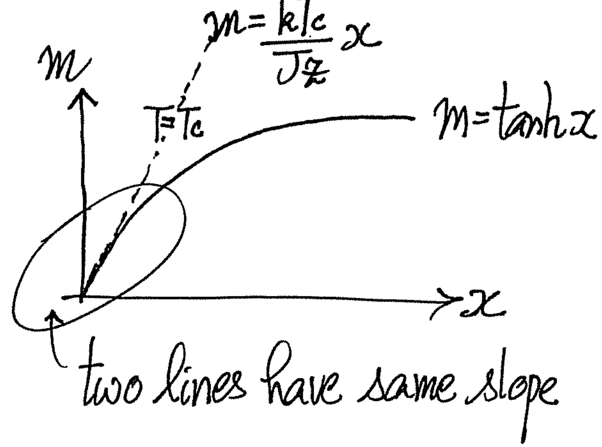


The physically realized solution(s) is (are) the one (ones) that gives (give) a minimum Helmholtz free energy.

∴ Yes! Mean-field theory predicts spontaneous magnetization for $T<T_c$.

- Critical temperature (MF prediction):

  At $T=T_c$, slopes at small x become the same.



∴ $m = \tanh x \approx x = \frac{Jz}{kT_c} m$

$$\Rightarrow \boxed{kT_c = \underset{\substack{\uparrow \\ \text{number of neighbors}}}{z} \underset{\substack{\uparrow \\ \text{strength of interaction}}}{J}} \quad \text{Mean-field } T_c^{(MF)} \qquad (6)$$

- Mean field Equation $m = \tanh\left(\frac{zJ}{kT} m\right)$ can be rewritten as

$$\boxed{m = \tanh\left(\frac{T_c}{T} \cdot m\right)} \qquad (7)$$

same as mean-field equation

- Mean field prediction on universal behavior (law of corresponding states)

$$m = \tanh\left(\frac{T_c}{T}\cdot m\right) \;\Rightarrow\; \underbrace{\frac{N}{V}\mu_B m}_{M} = \underbrace{\frac{N}{V}}_{M_s}\mu_B \tanh\left(\frac{T_c}{T}\cdot m\right)$$

putting back some constants

M magnetization

$$\Rightarrow \boxed{\frac{M}{M_s} = \tanh\left(\frac{T_c}{T}\cdot\frac{M}{M_s}\right)} \quad (8)$$

Mean field theory's suggestion on how to collapse data [Plot $\frac{M}{M_s}$ vs $\frac{T}{T_c}$]

- Mean-field theory gives reasonable results
- But often, mean-field theory does not give quantitatively accurate results [e.g. T_c is not correct, $m \sim (T_c - T)^\beta$ and β is not accurate]
- Mean-field theory is often the first thing to try in understanding an interacting system

Summary: Steps in setting up mean-field theory

- Decoupling the coupling term $S_i S_j \approx S_i \langle S_j \rangle$

  [Interacting system $\approx$ effective non-interacting system]

- Evaluating $\langle S \rangle$ using the approximated $E_{MF}(\{S_i\})$ to set up self-consistent equation(s)

- Same idea can be applied to many other problems